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Wielandt and Ky-Fan theorem for matrix pairs

Ivica Nakić [*,1], Krešimir Veselić

LG Mathematische Physik, Fernuniversität Hagen, Postfach 940, Hagen 58084, Germany

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Abstract

The generalization of Wielandt and Ky-Fan theorem is given for Hermitian matrix pairs, and some new eigenvalue perturbation estimates are obtained. An application is made on a class of quadratic matrix pencils.

Keywords: Matrix pencils; Quadratic pencils; Variational characterization of eigenvalues

1. Introduction

The classical Wielandt theorem states that the quantity

$$\sup_{\substack{\mathcal{M}_1\supset\cdots\supset\mathcal{M}_k\\ \dim\mathcal{M}_j=i_j}}\ \inf_{\substack{x_j\in\mathcal{M}_j\\ x_i^*x_j=\delta_{ij}}}\ \sum_{j=1}^{k} x_j^*Ax_j$$

equals the sum $\lambda_{i_1}+\cdots+\lambda_{i_k}$, where $\lambda_1\leqslant\cdots\leqslant\lambda_n$ are the eigenvalues of the Hermitian matrix $A\in\mathbb{C}^{n\times n}$, and $1\leqslant i_1<\cdots<i_k\leqslant n$ (see, for instance [16] and [2]).

The Wielandt theorem above was generalized to some specific matrix pairs. In fact, Svarzman [18] obtained a similar result for the class of Hermitian matrix pairs (A, J), where $J^{-1}=J$ and JA is a positive definite matrix. He obtained a three-part variational characterization of the form “inf sup inf” and “sup inf sup”.

Recently, Li and Mathias [13] proved a Wielandt-type theorem for definite matrix pairs (A, B), (i.e. pairs for which exist numbers $\alpha, \beta\in\mathbb{R}$ such that $\alpha A+\beta B$ is a

* Corresponding author.
E-mail addresses: nakic@math.hr (I. Nakić), kresimir.veselic@fernuni-hagen.de (K. Veselić).
[1] On leave on absence from Department of Mathematics, University of Zagreb.

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positive definite matrix), where, instead of eigenvalues, they considered the angles $\theta = \sphericalangle(Ax, Bx)$, with x an eigenvector of the pair (A, B).

The classical Ky-Fan theorem [2,16] states that

$$\min_{\substack{X \in \mathbb{C}^{n \times p} \\ X^* X = I_p}} \operatorname{Tr}(X^* A X)$$

equals the sum of the p smallest eigenvalues of the Hermitian matrix $A \in \mathbb{C}^{n \times n}$.

A generalization of this result to the class of Hermitian matrix pairs (A, B), where B is positive definite, was given in [15]. In [10], Kovač-Striko and Veselić obtained a Ky-Fan-type theorem for definite matrix pairs.

The classical Wielandt and Ky-Fan theorems are widely used in matrix theory. Using these theorems one can obtain various eigenvalue perturbation bounds, for instance, the famous Cauchy inequality (see [16])

$$\lambda_i(A) + \lambda_{\min}(E) \leqslant \lambda_i(A + E) \leqslant \lambda_i(A) + \lambda_{\max}(E)$$

can be derived from the Wielandt theorem. For other perturbation bounds derived from Wielandt and Ky-Fan theorems, see, for example, [2,13].

Our aim is to give a generalization of the Wielandt and Ky-Fan theorems for an arbitrary Hermitian matrix pair (A, B), the only assumption being the non-singularity of B.

The main tool in this paper will be the variational characterization of eigenvalues of Hermitian matrix pairs given in [6,7] (for a variational formulation in an infinite dimensional case see [4]).

In [7] the so-called "cancellation algorithm" was introduced. We show that we can variationally characterize, in the form "sup inf", the sum of those eigenvalues which "survive" a cancellation algorithm, and which do not "jump over" the cancelled pairs of eigenvalues. We first prove the theorem in the case when all real eigenvalues are semi-simple, the general case being a corollary of the semi-simple case. [2]

In our Ky-Fan-like theorem, we show that the sum of the p smallest eigenvalues of the same type can be variationally characterized. In both theorems, the matrix B induces a geometry in which we operate.

In the last section we apply both results to a "partially overdamped" quadratic eigenvalue problem by a convenient reduction of the quadratic matrix pencil to a linear pencil with Hermitian matrices.

2. Preliminaries

Let A, B be Hermitian matrices. We say that λ is an eigenvalue of the pair (A, B) if there exists $x \neq 0$ such that $Ax = \lambda Bx$. The set of all eigenvalues of (A, B) is denoted by $\sigma(A, B)$.

[2] In the first draft of the paper we only proved the semi-simple case. The referee's comments encouraged us to try to prove the general case, in which we succeeded.

In the rest of the paper we treat the case of a Hermitian matrix pair (A, B) from $\mathbb{C}^{N\times N}$, where B is non-singular.

For indefinite and non-singular B, it is often appropriate to work in the space with an indefinite inner product induced by B; see [9] for such an approach. Hence we introduce the inner product $[x, y] = y^*Bx$. A real eigenvalue λ is said to be of the positive (resp. negative) type if $[x, x] > 0$ (resp. < 0) for some eigenvector x associated with λ. Note that this definition differs from the usual one, where an eigenvalue λ is said to be of the positive (resp. negative) type if $[x, x] > 0$ (resp. < 0) for every eigenvector x associated with λ. Thus, it is possible that an eigenvalue λ is of both types. In this case it is always possible to find an eigenvector x such that $[x, x] = 0$. On the other hand, if we count multiplicities then a k-tuple semi-simple eigenvalue can be regarded as k eigenvalues, each of which is either of positive or of negative type. We denote the set of the eigenvalues of positive (resp. negative) type of the matrix pair (A, B) by $\sigma^+(A, B)$ (resp. $\sigma^-(A, B)$). By $n^+(A, B)$, $n^-(A, B)$ we denote the number of the eigenvalues of positive (resp. negative) type of pair (A, B), counting multiplicities.

Let $\{\pi(A), \nu(A), \delta(A)\}$ denote the inertia of a Hermitian matrix A, i.e. π, ν and δ are the number of positive, negative and zero eigenvalues of A, respectively.

By $\mathscr{S}_n$ we denote the sum of all eigenspaces corresponding to non-real eigenvalues.

For the sake of brevity, we introduce the following notation:
$a(x) = x^*Ax$, $b(x) = x^*Bx$, and $a \boxdot 0$ ($b \boxdot 0$) on some set $\mathscr{S}$ means $a(x) \boxdot 0$ ($b(x) \boxdot 0$) for all non-zero vectors from $\mathscr{S}$, where $\boxdot$ stands for the symbols $\geqslant, >, <, \leqslant$.

We say that a vector x is B-normalized if $b(x) = \pm 1$. We say that the vectors $\{x_1, \ldots, x_k\}$ form an B-orthonormal set if $x_i^*Bx_j = \pm\delta_{ij}$, where δ_{ij} denotes the Kronecker symbol. We say that the set $\{x_1, \ldots, x_k\}$ is a B-orthonormal basis for the subspace $\mathscr{W}$ if it is a basis for $\mathscr{W}$ and if it is B-orthonormal. By $\mathscr{B}_+$ ($\mathscr{B}_-$) we denote the set of all x such that $b(x) > 0$ (< 0).

3. Main results

First we consider the case when all real eigenvalues are semi-simple, i.e. with the corresponding Jordan chains of length one. As we have already said, in this case all real eigenvalues are either of positive or negative type.

Let $\lambda^-_{n^-} \leqslant \cdots \leqslant \lambda^-_1$ and $\lambda^+_1 \leqslant \cdots \leqslant \lambda^+_{n^+}$ be the eigenvalues of negative and positive type of the pair (A, B), respectively, counted by their multiplicity. If there exist $\lambda^\pm \in \sigma^\pm(A, B)$ such that $\lambda^+ < \lambda^-$, then we define γ_{-1} as the smallest λ^-_j greater then some λ^+_k, and γ_1 as the greatest $\lambda^+_j < \gamma_{-1}$. Obviously, $\gamma_{\pm 1}$ are well defined, the segment $[\gamma_1, \gamma_{-1}]$ is not empty and its interior (if not empty) contains no eigenvalues of (A, B). Now we repeat this procedure inductively on the set $\sigma(A, B) \setminus \{\gamma_{-i}, \gamma_{+i}\}$ until there are no $(+, -)$ pairs. The eigenvalues $\gamma_{\pm 1}, \ldots, \gamma_{\pm c}$ are called "cancelled" eigenvalues, and the number d given by

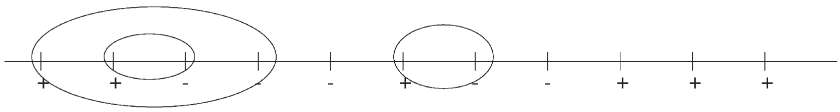

Fig. 1. Cancellation algorithm.

$$d = c + n, \quad n = \tfrac{1}{2}\dim(\mathscr{S}_n)$$

is called variational shift. This procedure is called the cancellation algorithm. By $r^{\pm} = r^{\pm}(A, B)$ we denote the number of non-cancelled eigenvalues of positive (resp. negative type). Notice that $N = 2c + 2n + r^+ + r^- = r^+ + r^- + 2d$. The following figure gives an example of the cancellation algorithm (Fig. 1).

Let $\mathscr{S}_c$ denote the sum of all eigenspaces corresponding to the cancelled eigenvalues.

The following theorem from [6] gives the variational characterization of the non-cancelled eigenvalues. Set

$$\sigma_k^{\pm} = \sup_{\substack{\mathscr{S} \\ \operatorname{codim}\mathscr{S}=k-1}} \inf_{\substack{x\in\mathscr{S} \\ b(x)\gtrless 0}} \frac{a(x)}{b(x)}, \tag{1}$$

with the convention $\inf\emptyset = -\infty$. We will state the theorem only for the eigenvalues of positive type. The formulation for the eigenvalues of negative type is analogous.

Theorem 1. *Let (A, B) be a Hermitian matrix pair such that all real eigenvalues are semi-simple. Let $\mu_1 \leqslant \cdots \leqslant \mu_{r^+}$ denote the non-cancelled eigenvalues of positive type. Then we have*

$$\sigma_k^{+} = \begin{cases} -\infty & \text{for } k \leqslant d, \\ \mu_{k-d} & \text{for } d < k \leqslant r^+ + d, \\ +\infty & \text{for } k > r^+ + d, \end{cases} \tag{2}$$

where d is the variational shift.

The numbers d and $r^+ + d$ can be obtained without prior knowledge of the spectrum. Indeed, it can be seen that $d = \min_{\lambda\in\mathbb{R}} \nu(\lambda)$ and $r^+ + d = \pi(B)$ (see [5]), where $\nu(\lambda) = \nu(A - \lambda B)$.

It is easy to prove that, instead of $\sigma_k^{\pm}$ as defined in formula (1), we can take

$$\sigma_k^{\pm} = \sup_{\substack{\mathscr{S} \\ \operatorname{codim}\mathscr{S}\leqslant k-1}} \inf_{\substack{x\in\mathscr{S} \\ b(x)\gtrless 0}} \frac{a(x)}{b(x)}. \tag{1'}$$

The same change can also be made in the case of the classical max–min formula for the eigenvalues of a Hermitian matrix.

Substituting A by $-A$ and/or B by $-B$, we get three other "dual" versions of Theorem 1. These "dual" variational formulations are obtained by interchanging inf and sup and/or and $\gtrless$ by $\lessgtr$ in (1) (and hence in (1′)), along with obvious modifications of the cancellation algorithm. This in general leads to different sets of non-cancelled eigenvalues.

3.1. A Wielandt-type theorem

We need the following two lemmas from [7].

Lemma 2. *For all real λ there exists a c-dimensional subspace $\mathscr{S}_c(\lambda) \subset \mathscr{S}_c$ on which $b > 0$ and $a - \lambda b < 0$ holds.*

Lemma 3. *For all real λ there exist n-dimensional subspaces $\mathscr{S}_{n0}$, $\mathscr{S}_{\pm}(\lambda) \subset \mathscr{S}_n$ such that*

$$a = b = 0 \quad \textit{on } \mathscr{S}_{n0}$$

and

$$b > 0, \ \pm(a - \lambda b) > 0 \quad \textit{on } \mathscr{S}_{\pm}(\lambda).$$

The following lemma is a simple generalization of Lemma 2.0 from [1].

Lemma 4. *Suppose that $\mathscr{W}_1 \supset \cdots \supset \mathscr{W}_k$ are subspaces of $\mathbb{C}^N$ such that $\dim \mathscr{W}_j = k - j + 1$, and $b > 0$ on $\mathscr{W}_j$, $j = 1, \ldots, k$. Let w_j, $j = 1, \ldots, k-1$ be B-orthonormal vectors such that $w_j \in \mathscr{W}_j$, $j = 1, \ldots, k-1$, and let $\mathscr{U} = \operatorname{span}\{w_1, \ldots, w_{k-1}\}$. Then there exists a vector $u \in \mathscr{W}_1 \setminus \mathscr{U}$, $u \neq 0$ such that $\mathscr{U} \dotplus \operatorname{span}\{u\}$ has a B-orthonormal basis $\{v_1, \ldots, v_k\}$, where $v_j \in \mathscr{W}_j$, $j = 1, \ldots, k$.*

Lemma 4, Theorem 2.1 and Corollary 2.2 from [1] imply the following result.

Lemma 5. *Let $\mathscr{N}_1 \subset \cdots \subset \mathscr{N}_k$ be subspaces such that $\dim \mathscr{N}_j = d + i_j$, $j = 1, \ldots, k$, $1 \leqslant i_1 < \cdots < i_k \leqslant N - d$, $0 \leqslant d < N$. Suppose also that $b > 0$ on $\mathscr{N}_j$, $j = 1, \ldots, k$. Let $\mathscr{M}_1 \supset \cdots \supset \mathscr{M}_k$ be another set of subspaces such that $\dim \mathscr{M}_j = N - (d + i_j) + 1$, $j = 1, \ldots, k$. Then there exists a B-orthonormal set $\{v_1, \ldots, v_k\}$ where $v_j \in \mathscr{N}_j$, $j = 1, \ldots, k$, and a B-orthonormal set $\{w_1, \ldots, w_k\}$ where $w_j \in \mathscr{M}_j$, $j = 1, \ldots, k$, such that*

$$\operatorname{span}\{v_1, \ldots, v_i\} = \operatorname{span}\{w_1, \ldots, w_i\}, \quad i = 1, \ldots, k.$$

Now we can state our generalization of Wielandt theorem for the matrix pairs, in the case of semi-simple real eigenvalues. The general case will be obtained as a corollary to this theorem.

Theorem 6. *Let (A, B) be a Hermitian matrix pair from $\mathbb{C}^{N \times N}$ with B non-singular and such that all real eigenvalues are semi-simple. We denote the non-cancelled eigenvalues of negative (resp. positive) type by $\mu^-_{r^-} \leqslant \cdots \leqslant \mu^-_1$ (resp. $\mu^+_1 \leqslant \cdots \leqslant \mu^+_{r^+}$), and the cancelled pairs by $\gamma_{\pm j} \in \sigma_{\pm}(A, B)$, $j = 1, \ldots, c$, $\gamma_1 < \cdots < \gamma_c$.*

(i) *Let $1 \leqslant i_1 < \cdots < i_k \leqslant r^+$. Let γ_{-j} be the greatest cancelled eigenvalue such that $\gamma_{-j} < \mu^+_{i_1}$ (if such an eigenvalue does not exist we take $j = 0$). Assume that*

$\mu^+_{i_k} < \gamma_{j+1}$ (*if such an eigenvalue does not exist this condition is void*), *i.e. that the eigenvalues* $\mu^+_{i_1}, \ldots, \mu^+_{i_k}$ *do not "jump over" the cancelled pairs* (γ_{-j}, γ_j). *Then*

$$\mu^+_{i_1} + \cdots + \mu^+_{i_k} = \sup_{\substack{\mathscr{N}_1 \supset \cdots \supset \mathscr{N}_k \\ \operatorname{codim} \mathscr{N}_j = i_j + d - 1}} \inf_{\substack{x_j \in \mathscr{N}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j, \tag{3}$$

where $d = c + n$ *is the variational shift.*

(ii) *Let* $1 \leqslant i_1 < \cdots < i_k \leqslant r^-$. *Let* γ_{-j} *be the greatest cancelled eigenvalue such that* $\gamma_{-j} < \mu^-_{i_k}$ (*if such an eigenvalue does not exist we take* $j = 0$). *Assume that* $\mu^-_{i_1} < \gamma_{j+1}$ (*if such an eigenvalue does not exist this condition is void*), *i.e. that the eigenvalues* $\mu^-_{i_k}, \ldots, \mu^-_{i_1}$ *do not "jump over" the cancelled pairs* (γ_{-j}, γ_j). *Then*

$$\mu^-_{i_1} + \cdots + \mu^-_{i_k} = -\inf_{\substack{\mathscr{N}_1 \supset \cdots \supset \mathscr{N}_k \\ \operatorname{codim} \mathscr{N}_j = i_j + d - 1}} \sup_{\substack{x_j \in \mathscr{N}_j \\ j=1,\ldots,k \\ x_j^* B x_i = -\delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j, \tag{4}$$

where $d = c + n$ *is the variational shift.*

Here we used convention $\inf \emptyset = -\infty$.

Proof. We will prove only (i), the other statement can be proved by applying the statement (i) to the matrix pair $(A, -B)$.

By $u^\pm_j$, $u_{\pm j}$, we denote the B-orthonormal eigenvectors corresponding to $\mu^\pm_j$ and $\gamma_{\pm j}$, respectively.

We define $z_l = u_{\pm l}$ such that $\pm(\gamma_{\pm l} - \mu^+_{i_j}) \geqslant 0$, $1 \leqslant j \leqslant k$. Set

$$\mathscr{V}_j = \mathscr{S}_n \dotplus \operatorname{span}\{u^+_{i_j}, \ldots, u^+_{r^+}, u^-_1, \ldots, u^-_{r^-}, z_1, \ldots, z_c\}. \tag{5}$$

Obviously, $\mathscr{V}_1 \supset \cdots \supset \mathscr{V}_k$ and $\dim \mathscr{V}_j = N - (d + i_j) + 1$. Note that the "jumping over" assumption was used here.

From a straightforward calculation it follows that $a - \mu^+_{i_j} b \geqslant 0$ on $\mathscr{V}_j$, and $(a - \mu^+_{i_j} b)(u^+_{i_j}) = 0$, hence

$$\min_{\substack{x_j \in \mathscr{V}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} a(x_j) = \sum_{j=1}^{k} \mu^+_{i_j}.$$

This implies

$$\sup_{\substack{\mathscr{V}_1 \supset \cdots \supset \mathscr{V}_k \\ \operatorname{codim} \mathscr{V}_j = i_j + d - 1}} \inf_{\substack{x_j \in \mathscr{V}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} a(x_j) \geqslant \sum_{j=1}^{k} \mu^+_{i_j}. \tag{6}$$

On the other hand, let $\mathscr{M}_1 \supset \cdots \supset \mathscr{M}_k$ be subspaces such that $\operatorname{codim} \mathscr{M}_j = d + i_j - 1$. To prove the relation (3) it is sufficient to find vectors $x_j \in \mathscr{M}_j$, $j = 1, \ldots, k$, $x_i^* B x_j = \delta_{ij}$ such that $\sum_{j=1}^k a(x_j) \leqslant \sum_{j=1}^k \mu_{i_j}^+$. Finally, we define

$$\mathscr{N}_j = \mathscr{S}_c(\mu_{i_1}^+) \dot{+} \mathscr{S}_-(\mu_{i_1}^+) \dot{+} \operatorname{span}\{u_1^+, \ldots, u_{i_j}^+\}.$$

Obviously, $\mathscr{N}_1 \subset \cdots \subset \mathscr{N}_k$ and $\dim \mathscr{N}_j = d + i_j$. It is easy to see that $b > 0$ on $\mathscr{N}_j$ and that $a - \mu_{i_j}^+ b \leqslant 0$ on $\mathscr{N}_j$. Now, Lemma 5 implies that there exist a B-orthonormal sets $\{x_1, \ldots, x_k\}$, with $x_j \in \mathscr{M}_j$, $j = 1, \ldots, k$ and $\{y_1, \ldots, y_k\}$, $y_j \in \mathscr{N}_j$, $j = 1, \ldots, k$, such that

$$\operatorname{span}\{x_1, \ldots, x_i\} = \operatorname{span}\{y_1, \ldots, y_i\}, \quad i = 1, \ldots, k.$$

Let us introduce the subspace

$$\mathscr{W} = \operatorname{span}\{x_1, \ldots, x_k\}\big(= \operatorname{span}\{y_1, \ldots, y_k\}\big).$$

Since $\mathscr{W} \subset \mathscr{N}_k$, the subspace $\mathscr{W}$, together with the scalar product defined by $[x, y] = y^* B x$, can be regarded as a Hilbert space. By $\mathscr{W}^{[\perp]}$ we denote the orthogonal complement of $\mathscr{W}$ generated by this scalar product, i.e.

$$\mathscr{W}^{[\perp]} = \{x \in \mathbb{C}^N : [x, y] = 0, \forall y \in \mathscr{W}\}.$$

Since $\mathscr{W}$ is a Hilbert space with this scalar product, $\mathscr{W}^{[\perp]}$ is well-defined. By $P_{\mathscr{W}}$ we denote the projector on $\mathscr{W}$ along the subspace $\mathscr{W}^{[\perp]}$. Set $Q_{\mathscr{W}} = P_{\mathscr{W}} B^{-1} A x$, for $x \in \mathscr{W}$. Then

$$[Q_{\mathscr{W}} x, y] = x^* A y, \quad \text{for all } x, y \in \mathscr{W},$$

hence $Q_{\mathscr{W}}$ is a Hermitian operator on $(\mathscr{W}, [\cdot, \cdot])$. Then $\operatorname{Tr} Q_{\mathscr{W}} = \sum_{j=1}^k \tau_j$, where $\tau_1 \leqslant \cdots \leqslant \tau_k$ are the eigenvalues of $Q_{\mathscr{W}}$.

Let v_j be the eigenvector corresponding to τ_j, $j = 1, \ldots, k$. Set $\mathscr{U}_j = \operatorname{span}\{x_1, \ldots, x_j\}$, $\mathscr{V}_j = \operatorname{span}\{v_j, \ldots, v_k\}$. Then $\mathscr{U}_j, \mathscr{V}_j \subset \mathscr{W}$ and $\mathscr{U}_j \cap \mathscr{V}_j \neq \{0\}$. Let $u_j \in \mathscr{U}_j \cap \mathscr{V}_j$ be such that $b(u_j) = 1$. Note that $\operatorname{span}\{x_1, \ldots, x_j\} = \operatorname{span}\{y_1, \ldots, y_j\}$, hence $u_j \in \mathscr{N}_j$, and $a(u_j) \leqslant \mu_{i_j}^+$. Then

$$\tau_j \leqslant [Q_{\mathscr{W}} u_j, u_j] = a(u_j) \leqslant \mu_{i_j}^+, \quad j = 1, \ldots, k.$$

Using this we obtain

$$\begin{aligned}\sum_{j=1}^k a(x_j) &= \sum_{j=1}^k [Q_{\mathscr{W}} x_j, x_j] = \operatorname{Tr} Q_{\mathscr{W}} \\ &= \sum_{j=1}^k \tau_j \leqslant \sum_{j=1}^k \mu_{i_j}^+,\end{aligned}$$

which together with (6), finishes the proof. □

The general case will be treated using a perturbation technique. From [6], it always follows the existence of a number $\delta_0 > 0$, and a positive semi-definite matrix

P, such that the pair $(A + \delta P, B)$ has all real eigenvalues semi-simple, for all $0 < \delta < \delta_0$. This perturbation also has the following property: the real eigenvalue of the pair (A, B) with Jordan chain of length $m > 1$ and of the type $* \in \{+, -\}$ under perturbation becomes

(i) one semi-simple eigenvalue of the type $*$ and k conjugate pairs of non-real eigenvalues, if $m = 2k + 1$,
(ii) two semi-simple eigenvalues, one of the $+$ type, the other of the $-$ type, where the eigenvalue of negative type is smaller of the eigenvalue of positive type, and $k - 1$ conjugate pairs of non-real eigenvalues, if $m = 2k$ and $*$ equals $+$,
(iii) k conjugate pairs of non-real eigenvalues, if $m = 2k$ and $*$ equals $-$.

Now we introduce the extended cancellation algorithm (see [6]). First we perturb the matrix A by δP, with $0 < \delta < \delta_0$, and then use the cancellation algorithm defined before.

We have [6]

$$d(\delta) = d, \quad \text{for all } 0 < \delta < \delta_0,$$

where $d(\delta) = d(A + \delta P, B)$. We will state the result only for the eigenvalues of positive type, the version for the eigenvalues of negative type is similar.

Theorem 7. *Let (A, B) be a Hermitian matrix pair from $\mathbb{C}^{N\times N}$, with B non-singular. We denote the non-cancelled eigenvalues of negative (resp. positive) type of extended cancellation algorithm by $\mu^-_{r^-} \leqslant \cdots \leqslant \mu^-_1$ (resp. $\mu^+_1 \leqslant \cdots \leqslant \mu^+_{r^+}$), and the cancelled pairs by $\gamma_{\pm j} \in \sigma_\pm(A, B)$, $j = 1, \ldots, c$, $\gamma_1 < \cdots < \gamma_c$.*

Let $d = n + c$, where $2n$ is the dimension of the eigenspace corresponding to all non-real eigenvalues and of those eigenvalues which become non-real under the perturbation.

Let $1 \leqslant i_1 < \cdots < i_k \leqslant r^+$. Let γ_{-j} be the greatest cancelled eigenvalue such that $\gamma_{-j} < \mu^+_{i_1}$ (if such an eigenvalue does not exist we take $j = 0$). Assume that $\mu^+_{i_k} < \gamma_{j+1}$ (if such an eigenvalue does not exist this condition is void), i.e. that eigenvalues $\mu^+_{i_1}, \ldots, \mu^+_{i_k}$ do not "jump over" the cancelled pairs (γ_{-j}, γ_j).

Then (3) holds.

Proof. Let us denote with $\mu^+_i(\delta)$ the corresponding perturbed eigenvalues of the pair $(A + \delta P, B)$, $0 < \delta < \delta_0$. Since the eigenvalues $\mu^+_{i_1}(\delta), \ldots, \mu^+_{i_k}(\delta)$ satisfy the assumptions of the Theorem 6, we have

$$\mu^+_{i_1}(\delta) + \cdots + \mu^+_{i_k}(\delta) = \sup_{\substack{\mathscr{N}_1 \supset \cdots \supset \mathscr{N}_k \\ \operatorname{codim} \mathscr{N}_j = i_j + d(\delta) - 1}} \inf_{\substack{x_j \in \mathscr{N}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} x_j^* (A + \delta P) x_j.$$

Now, when $\delta \searrow 0$ we obtain (3). $\square$

By simple substitution of A by $-A$ and/or B by $-B$ we get three other "dual" versions of the cancellation algorithm, and, hence, of Lemmas 2, 3 and of Theorem 7. Also, by imitating the proof of [1, Theorem 2.3], we can generalize Theorem 7 by replacing the left hand side of (3) by $\varphi(\mu^+_{i_1}, \dots, \mu^+_{i_k})$, and the right hand side by the corresponding relation, where φ is some symmetric function in k variables. We omit the details.

Note that for definite matrix pairs $d = 0$, hence all eigenvalues are non-cancelled. Also, from the proof of Theorem 6 it follows that the inequality

$$\mu^+_{i_1} + \cdots + \mu^+_{i_k} \geqslant \sup_{\substack{\mathscr{N}_1 \supset \cdots \supset \mathscr{N}_k \\ \operatorname{codim} \mathscr{N}_j = i_j + d}} \inf_{\substack{x_j \in \mathscr{N}_j \\ j=1,\dots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j,$$

holds regardless of the "jumping over" condition.

We conjecture that the "jumping over" assumption cannot in general be discarded, but we were not able to find an appropriate example.

Remark 8. Let (A, B) be a strongly definitizable matrix pair, i.e. a Hermitian matrix pair with all real eigenvalues semi-simple, and not of mixed type (for more details see [11,14]). Let δA be a Hermitian perturbation such that the pair $(\widetilde{A}, B) = (A + \delta A, B)$ has the same eigenvalue structure as the pair (A, B). The sufficient conditions on δA which imply these assumptions can be found in [14].

Using the technique of [17] (see also [12]) we can obtain an eigenvalue bound. We present only the bound for the eigenvalues of positive type, bound for the eigenvalues of negative type is similar.

Let μ^+_j, $1 \leqslant j \leqslant r^+$, be the non-cancelled eigenvalue of positive type, where the eigenvalues are ordered as in Theorem 6, and let $\widetilde{\mu}^+_j$ be the corresponding eigenvalue of the pair $(\widetilde{A}, B)$. Let $\mathscr{V}_j$ be defined by (5), and let $\widetilde{\mathscr{V}}_j$ be a corresponding subspace for the pair $(\widetilde{A}, B)$.

We define

$$r_j = \max \left\{ \sup_{\substack{x \in \mathscr{V}_j \\ b(x)=1}} x^* \delta A x, \ \sup_{\substack{x \in \widetilde{\mathscr{V}}_j \\ b(x)=1}} x^* \delta A x \right\}.$$

Then

$$\left| \widetilde{\mu}^+_j - \mu^+_j \right| \leqslant r_j.$$

Obviously, r_j can be bounded with $\eta = \sup_{b(x)=1} x^* \delta A x$, but, as can be seen from (2), it is possible that $\eta = \infty$.

3.2. Ky-Fan theorem

Now we prove the generalization of the well-known Ky-Fan theorem. The theorem will be formulated only for the eigenvalues of positive type, the formulation for the eigenvalues of negative type being analogous.

Theorem 9. *Let (A, B) be a Hermitian matrix pair from $\mathbb{C}^{N\times N}$, where B is non-singular. Let $\mu_j^{\pm}$, $r^{\pm}$, and c be defined as in Theorem 7. Then for $1 \leqslant p \leqslant r^+$ we have*

$$\min_{\substack{X\in\mathbb{C}^{N\times p}\\ X^*BX=I_p}} \operatorname{Tr}(X^*AX) = \sum_{j=1}^{p} \mu_i^+, \tag{7}$$

where I_p denotes the identity matrix in $\mathbb{C}^{p\times p}$.

Proof. As before, we first treat the case when all real eigenvalues are semi-simple. Set $X = [x_1 \cdots x_p]$, where x_i are the eigenvectors corresponding to μ_i^+, $i = 1, \ldots, p$, such that $b(x_i) = 1$. Then, obviously,

$$\operatorname{Tr}(X^*AX) = \sum_{i=1}^{p} \mu_i^+ \quad \text{and} \quad X^*BX = I_p.$$

Hence, we have proved

$$\min_{\substack{X\in\mathbb{C}^{N\times p}\\ X^*BX=I_p}} \operatorname{Tr}(X^*AX) \leqslant \sum_{j=1}^{p} \mu_i^+.$$

Now, let $X \in \mathbb{C}^{N\times p}$ be arbitrary such that $X^*BX = I_p$. By $\mathscr{X}$ we denote the subspace in $\mathbb{C}^N$ spanned by the columns of X. Obviously, $f^*Bf > 0$ for each non-zero vector f from $\mathscr{X}$.

By $\tau_1 \leqslant \cdots \leqslant \tau_p$ we denote the eigenvalues of the matrix X^*AX. From previous considerations and the variational principle given in Theorem 1, for $1 \leqslant i \leqslant p$ it follows

$$\begin{aligned}
\tau_i &= \max_{\substack{\mathscr{S}\subset\mathbb{C}^p\\ \dim S\leqslant i-1}} \min_{\substack{f\in\mathbb{C}^p\\ f^*g=0,\\ g\in\mathscr{S}}} \frac{f^*X^*AXf}{f^*f} = \max_{g_1,\ldots,g_{i-1}\in\mathbb{C}^p} \min_{\substack{f\in\mathbb{C}^p\\ (Xf)^*B(Xg_j)=0,\\ j=1,\ldots,i-1}} \frac{(Xf)^*A(Xf)}{(Xf)^*B(Xf)}\\
&= \max_{g_1,\ldots,g_{i-1}\in\mathscr{X}} \min_{\substack{f\in\mathscr{X}\\ f^*Bg_j=0,\\ j=1,\ldots,i-1}} \frac{f^*Af}{f^*Bf} = \max_{g_1,\ldots,g_{i-1}\in\mathbb{C}^N} \min_{\substack{f\in\mathscr{X}\\ f^*Bg_j=0,\\ j=1,\ldots,i-1}} \frac{f^*Af}{f^*Bf}\\
&\geqslant \sup_{g_1,\ldots,g_{i-1}\in\mathbb{C}^N} \inf_{\substack{f\in\mathbb{C}^N\\ f^*Bg_j=0,\\ j=1,\ldots,i-1\\ b(f)>0}} \frac{a(f)}{b(f)} = \sup_{\substack{\mathscr{M}\subset\mathbb{C}^N\\ \operatorname{codim}\mathscr{M}\leqslant i-1}} \inf_{\substack{f\in\mathscr{M}\\ b(f)>0}} \frac{a(f)}{b(f)} = \mu_i^+.
\end{aligned}$$

Since $\tau_1 + \cdots + \tau_p = \operatorname{Tr} X^*AX$, follows

$$\operatorname{Tr} X^*AX \geqslant \sum_{j=1}^{p} \mu_j^+.$$

Hence, we have proved the Theorem in the semi-simple case.

As before, by use of the perturbation technique, we easily prove the general case. □

4. An application to quadratic pencils

In this section we apply obtained results to a quadratic matrix pencil

$$L(\lambda) = \lambda^2 I + \lambda C + K,$$

where C, K are symmetric matrices of order N such that K is positive definite and C is positive semi-definite.

The spectrum of the matrix pencil $L(\lambda)$ is the set of complex numbers λ such that there is a non-zero vector x such that $L(\lambda)x = 0$. It is easy to see that the spectrum of $L(\lambda)$ is situated in the left half plane. We denote the spectrum of L by $\sigma(L)$.

We now describe the class of pencils which will be treating.

We assume that matrices C and K can be decomposed in the following way:

$$C = \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix}, \tag{8}$$

where C_{22} is positive definite. We assume also that the operator pencil $L_2(\lambda) = \lambda^2 I + \lambda C_{22} + K_{22}$ is overdamped. (A quadratic pencil $L(\lambda) = \lambda^2 + \lambda C + K$ is overdamped if $(x^*Cx)^2 > 4x^*Kx$ holds for all $\|x\| = 1$.)

Next we introduce following block-matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & K^{1/2} \\ K^{1/2} & C \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{9}$$

It is easy to see that λ is an eigenvalue of the matrix pencil $L(\lambda)$ if and only if λ is an eigenvalue of the Hermitian matrix pair $(\mathbf{A}, \mathbf{B})$. So, we can introduce the $\mathbf{B}$-sign characteristics of the eigenvalues of the matrix pencil $L(\lambda)$, as is done in Section 2.

To clarify the statements, we will use the following convention. Vectors corresponding to the pencil L_2, i.e. vectors which are applied to the matrices C_{22}, K_{22} are denoted by letters u and v. Vectors which are applied to the matrices C and K are denoted by letters x and y, and vectors which are applied to the matrices $\mathbf{A}$ and $\mathbf{B}$ are denoted by the bold font.

Our aim in this section is to prove the following theorem.

Theorem 10. *Let $L(\lambda) = \lambda^2 I + \lambda C + K$ be a Hermitian quadratic pencil of order N with C positive semi-definite and K positive definite. Assume that*

(1) *matrices C and K can be decomposed as in* (8), *and*
(2) *the operator pencil $L_2(\lambda) = \lambda^2 I + \lambda C_{22} + K_{22}$ is overdamped.*

Let r be the size of C_{22}. Then there exists a gap in the real spectrum of the pencil $L(\lambda)$, i.e. there exist a segment $[\alpha, \beta]$, $\alpha < \beta$ such that $[\alpha, \beta] \cap \sigma(L) \cap \mathbb{R} = \emptyset$, and there exist r real eigenvalues on the left side of the gap. Let us denote them by $\mu_r^- \leqslant \cdots \leqslant \mu_1^-$. Then the following holds.

1. *Let $1 \leqslant i_1 < \cdots < i_k \leqslant r$. Then*

$$\mu_{i_1}^- + \cdots + \mu_{i_k}^- = - \inf_{\substack{\mathscr{N}_1 \supset \cdots \supset \mathscr{N}_k \\ \operatorname{codim} \mathscr{N}_j = i_j + d - 1}} \sup_{\substack{\mathbf{x}_j \in \mathscr{N}_j \\ j=1,\ldots,k \\ \mathbf{x}_j^* \mathbf{B} \mathbf{x}_i = -\delta_{ij}}} \sum_{j=1}^{k} \mathbf{x}_j^* \mathbf{A} \mathbf{x}_j \tag{10}$$

holds, where d is the variational shift for the matrix pair $(\mathbf{A}, \mathbf{B})$.

2. *We have*

$$\sum_{j=1}^{r} \mu_i^- = - \max_{\substack{\mathbf{X} \in \mathbb{C}^{2N \times r} \\ \mathbf{X}^* \mathbf{B} \mathbf{X} = -I_r}} \operatorname{Tr}(\mathbf{X}^* \mathbf{A} \mathbf{X}).$$

Proof

First we prove the existence of a gap in the real spectrum of L.

Let $x = \binom{u}{v}$. Then

$$(\lambda^2 I + \lambda C + K)x = 0$$

can be written as

$$\lambda^2 \begin{bmatrix} u \\ v \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

From this follows

$$\lambda^2 u + K_{11} u + K_{12} v = 0,$$
$$\lambda^2 v + \lambda C_{22} v + K_{12}^* u + K_{22} v = 0.$$

Substituting u in the second equation, from the first we get

$$\lambda^2 v + \lambda C_{22} v + (K_{22} - K_{12}^* (K_{11} + \lambda^2)^{-1} K_{12}) v = 0. \tag{11}$$

We introduce the Hermitian matrix function

$$K(\lambda) = K_{22} - K_{12}^* (K_{11} + \lambda^2)^{-1} K_{12}, \quad \lambda \in \mathbb{R},$$

and the (non-linear) functionals

$$p_{\pm}(v, \lambda) = \frac{-v^* C_{22} v \pm \sqrt{(v^* C_{22} v)^2 - 4 v^* v v^* K(\lambda) v}}{2 v^* v}. \tag{12}$$

Obviously, $p_\pm(v,\lambda)=\lambda$, if and only if $\lambda^2 v^*v+\lambda v^*C_{22}v+v^*K(\lambda)v=0$. Hence, if λ is an eigenvalue of $L(\lambda)$ with the corresponding eigenvector $x=\binom{u}{v}$, then we have $p_+(v,\lambda)=\lambda$ or $p_-(v,\lambda)=\lambda$ or both. Since $K(\lambda)\leqslant K_{22}$ for real λ, we have $p_+(v,\lambda)\geqslant p'_+(v)$ and $p_-(v,\lambda)\leqslant p'_-(v)$, where $p'_\pm$ is the corresponding functional for the matrix pencil $L_2(\lambda)$, given by

$$p'_\pm(v)=\frac{-v^*C_{22}v\pm\sqrt{(v^*C_{22}v)^2-4v^*vv^*K_{22}v}}{2v^*v}.$$

Since the matrix pencil $L_2(\lambda)$ is overdamped, we have $\sup p'_- < \inf p'_+$ [8], hence it follows that there is a gap in the real spectrum of the pencil $L(\lambda)$, and we can take $\alpha=\sup p'_-$, $\beta=\inf p'_+$.

We will show that each real eigenvalue of the left hand side of the gap is of negative type, and therefore all real eigenvalues on the left hand side of the gap are semi-simple.

Let (λ,x), $\lambda\in\mathbb{R}$, $\lambda<\alpha$ be an eigenpair of the matrix pencil $L(\lambda)$, $x=\binom{u}{v}$. Then $p_-(v,\lambda)=\lambda$, i.e.

$$\lambda=\frac{-v^*C_{22}v-\sqrt{(v^*C_{22}v)^2-4v^*vv^*K(\lambda)v}}{2v^*v}<-\frac{v^*C_{22}v}{2v^*v}.$$

Since $v^*v\leqslant x^*x$ and $v^*C_{22}v=x^*Cx$, it follows

$$\lambda<-\frac{x^*Cx}{2x^*x}.$$

This implies (λ is necessarily negative!)

$$2\lambda^2x^*x+\lambda x^*Cx>0,$$

which together with

$$\lambda^2x^*x+\lambda x^*Cx+x^*Kx=0$$

implies

$$\lambda^2x^*x>x^*Kx. \tag{13}$$

Set $x_1=(1/\lambda)K^{1/2}x$. Then one can easily check that $\mathbf{x}=\binom{x_1}{x}$ is an eigenvector of the matrix pair $(\mathbf{A},\mathbf{B})$ corresponding to the eigenvalue λ.

Now (13) implies $x_1^*x_1<x^*x$, hence $\mathbf{x}^*\mathbf{B}\mathbf{x}<0$.

Next we show that the number of real eigenvalues on the left hand side of the gap, counted by multiplicity, is precisely r. The number of these left eigenvalues is determined by the following continuity argument. Consider the family of pencils

$$L(\lambda,\varepsilon)=\lambda^2+\lambda\varepsilon C+K,\quad \varepsilon\geqslant 0.$$

For any $\varepsilon\geqslant 1$ this family obviously satisfies the conditions of our theorem. The negative spectral gap increases with ε, more precisely, the corresponding functionals $p_-(v,\lambda,\varepsilon)$ and $p_+(v,\lambda,\varepsilon)$ satisfy

$$p_-(v,\lambda,\varepsilon) \leqslant p'_-(v,\varepsilon) = -\frac{\varepsilon v^* C_{22} v}{2v^*v}\left(1+\sqrt{1-\frac{4v^*vv^*K_{22}v}{(\varepsilon v^*C_{22}v)^2}}\right)$$
$$\leqslant -\frac{\varepsilon v^* C_{22} v}{2v^*v}\left(1+\sqrt{1-\frac{\delta}{\varepsilon^2}}\right),$$

$$p_+(v,\lambda,\varepsilon) \geqslant p'_+(v,\varepsilon) \geqslant -\frac{\varepsilon v^* C_{22} v}{2v^*v}\left(1-\sqrt{1-\frac{\delta}{\varepsilon^2}}\right),$$

with

$$\delta = \sup_{\substack{v\in\mathbb{R}^r\\ v\neq 0}} \frac{4v^*vv^*K_{22}v}{(v^*C_{22}v)^2}.$$

Since the pencil $L_2(\lambda)$ is overdamped, it follows $\delta < 1$.

The left eigenvalues are real analytic functions of ε without singularities—even if the eigenvalues should cross themselves. This is because they are of positive type (a proof can be found e.g. in [19]). This precludes any mixing with possible non-real eigenvalues. Thus, as $\varepsilon \nearrow \infty$ these eigenvalues tend to $-\infty$ and do not change their number, the multiplicities included.

On the other hand, no other eigenvalues can approach infinity from any (complex) direction. Indeed, let us assume that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of $L(\lambda)$ with the corresponding eigenvector x. Then

$$\lambda = \frac{-x^*Cx \pm \sqrt{(x^*Cx)^2 - 4x^*xx^*Kx}}{2x^*x}.$$

Since λ is non-real, this can be written as

$$\lambda = \frac{-x^*Cx}{2x^*x} \pm \mathrm{i}\frac{\sqrt{4x^*xx^*Kx - (x^*Cx)^2}}{2x^*x},$$

so we have

$$|\lambda|^2 = \frac{x^*Kx}{x^*x}. \tag{14}$$

Hence

$$\|K^{-1}\|^{-1} \leqslant |\lambda|^2 \leqslant \|K\|. \tag{15}$$

Thus, the number of the eigenvalues in question is equal to the dimension of the spectral subspace of the matrix

$$\lim_{\varepsilon\to\infty} (\mathbf{BA}(\varepsilon) - \lambda)^{-1} \tag{16}$$

belonging to the eigenvalue zero. Here λ is a fixed real number between $\sup p'_-$ and $\inf p'_+$ and $\mathbf{A}(\varepsilon)$ is the corresponding block matrix. We compute the limit (16). We calculate

$$\lim_{\varepsilon\to\infty} L(\lambda,\varepsilon)^{-1} = \begin{bmatrix} (\lambda^2+K_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Now the well-known formula

$$(\mathbf{BA}(\varepsilon)-\lambda)^{-1} = \begin{bmatrix} -\frac{1}{\lambda}+\frac{1}{\lambda}K^{1/2}L(\lambda,\varepsilon)^{-1}K^{1/2} & -K^{1/2}L(\lambda,\varepsilon)^{-1} \\ L(\lambda,\varepsilon)^{-1}K^{1/2} & -\lambda L(\lambda,\varepsilon)^{-1} \end{bmatrix}$$

implies

$$\lim_{\varepsilon\to\infty}(\mathbf{BA}(\varepsilon)-\lambda)^{-1} = \begin{bmatrix} -\frac{1}{\lambda}+\frac{1}{\lambda}K^{1/2}\begin{bmatrix} (\lambda^2+K_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix}K^{1/2} & -K^{1/2}\begin{bmatrix} (\lambda^2+K_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} (\lambda^2+K_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix}K^{1/2} & -\lambda\begin{bmatrix} (\lambda^2+K_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

One can easily see that the defect of this block-matrix is exactly r.

Hence, we can apply negative-type analogues of Theorems 7 and 9 to the matrix pair $(\mathbf{A},\mathbf{B})$ to obtain desired result. □

From the proof of the foregoing theorem it follows that everything said will hold for the positive-type eigenvalues as well, provided that

$$\inf_v p'_+(v) < \|K\|^{-1/2}$$

(we omit the details).

Unfortunately, the real eigenvalues on the right hand side of the gap need not be of positive type. This can be readily seen from Fig. 2. The picture in Fig. 2 is obtained by plotting the eigenvalues of random matrix pencils of order three, which satisfy our conditions, while keeping the smallest eigenvalue of K to one.

A concrete example is the matrix pencil with the following entries:

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 301.822 \end{bmatrix}, \quad K = \begin{bmatrix} 136.63382 & 113.58142 & 1583.7435 \\ 113.58142 & 97.472318 & 1343.3431 \\ 1583.7435 & 1343.3431 & 18709.024 \end{bmatrix}.$$

The eigenvalues of this pencil are $\lambda_1 = -215.39099$, $\lambda_2 = -82.517913$, $\lambda_3 = -3.3700361$, $\lambda_{4,5} = -0.04725 \pm 1.34\mathrm{i}$, $\lambda_6 = -0.44885985$. The eigenvalues λ_1 and λ_3 are of negative type, other (real) eigenvalues are of positive type. The gap is $[-214.6697,\ -87.152]$, hence the eigenvalue λ_3 is on the right hand side of the gap.

Moreover, inside of the circle with the center in the origin and the radius $\lambda_{\min}^{1/2}(K)$ there are no non-real eigenvalues, which follows from (14).

One could show that all eigenvalues which are contained in this circle are of positive type and that Theorems 6 and 9 are applicable for them, too. We omit the details.

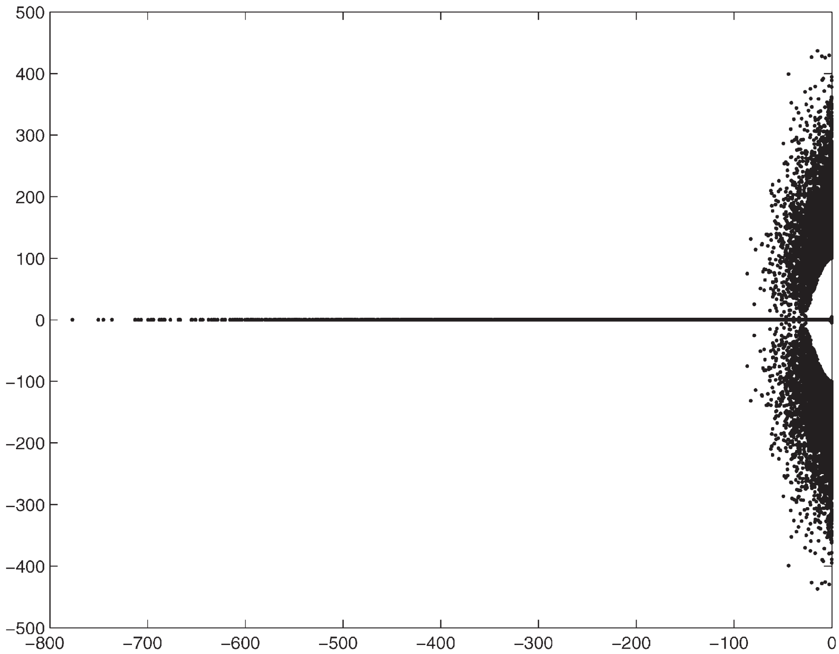

Fig. 2. Distribution of the eigenvalues.

Another characterization of a single eigenvalue of the negative type was given in [3]; it reads

$$\mu_j^- = \min_{\substack{S \\ \dim S = j \\ S \subset \mathscr{M}}} \max_{\|x\|=1} p_-(x),$$

where $\mathscr{M} = \{x : (x^*Cx)^2 \geqslant x^*Kxx^*x\}$, and p_- is given by

$$p_-(x) = \frac{-x^*Cx - \sqrt{(x^*Cx)^2 - 4x^*xx^*Kx}}{2x^*x}.$$

Our formula (10), specialized for the case of a single eigenvalue reads

$$\mu_j^- = -\inf_{\operatorname{codim} \mathscr{N} = j+d-1} \sup_{\substack{\mathbf{x} \in \mathscr{N} \\ \mathbf{x}^*\mathbf{B}\mathbf{x} = -1}} \mathbf{x}^*\mathbf{A}\mathbf{x},$$

where **A** and **B** are given in (9).

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